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Approximate Solution of Space Fractional Partial Differential Equations and Its Applications

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Abstract: **- The purpose of this paper is to obtain the solution of space fractional partial differential equations by Adomian Decomposition Method (ADM). The solutions are derived in convergent series form which shows the effectiveness of method for solving wide variety of fractional differential equations. We obtain the solution of some test problems and these are graphically represented by Mathematica.**

Index Terms**— Adomian Decomposition Method, Caputo fractional derivative, Mathematica, Space fractional differential equation.**

I. INTRODUCTION

In 1980, Adomian proposed decomposition method known as Adomian Decomposition Method (ADM) [7, 8, 9]. Over last twenty years, the ADM has been applied to a wide class of deterministic and stochastic partial differential equations. This method has been intensively applied by researchers because; it provides analytical approximate solution for nonlinear partial differential equations without linearization, perturbation, and discretization. The convergence of ADM has been solved by many authors. A comparison between ADM and Taylor series method to the solution of linear and nonlinear Ordinary Differential Equation is contributed by A.M.Wazwaz in his work [2]. For a better understanding of the fractional derivatives and for a physical understanding of the fractional equations, the readers can refer the research paper written by Jin-Fa Cheng and Yu-Ming Chu [11]. Mainardi [4], [5], [6] presented analytical investigation of the time fractional diffusion wave equations. He also provided a comprehensive review of research on the application of calculus in continuum and statistical mechanics including research on fractional diffusion-wave solutions. Agrawal [16] presented a general solution for a time fractional diffusion-wave equation defined in a bounded space domain. Al-Khaled and Momani [12] used the ADM to obtain an approximate solution for the generalized time-fractional diffusion-wave equation. Researcher like Duan Junsheng, An Sianye and Xu Mingyu have applied ADM to solve time fractional partial differential equation by ADM [3]. Also Shaher Momani has solved space time fractional diffusion wave equation by ADM [17]. Mridula Garg and Ajay Sharma worked for solution of space-time fractional Telegraph equation by ADM [14]. Recently, ordinary and partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, biology, physics and engineering. Consequently, considerable attention has been given to the solutions of fractional partial differential equations of physical interest [3]. For better understanding of a phenomenon described by a given nonlinear fractional partial differential equation, the solutions of differential equations of fractional order are much involved. Fractional derivatives provide more accurate models of real world problems than integer order derivatives. Because of their many applications in scientific fields, fractional partial differential equations are found to be an effective tool to describe certain physical phenomenon such as diffusion process, electrical and rheological material properties, control science, electro magnetic theory, capacitor theory, vibrating damping system, hereditary prediction of gene behaviour, fractional neural modelling on bio-sciences, communication channel traffic models, viscoelasticity theories and several more [1], [10], [13]. This brief review of fractional diffusion equation encouraged us to work in fractional calculus. In our work, we developed the most general Fractional Adomian Decomposition Method (FADM) for linear and nonlinear space fractional diffusion equation in time direction. As an application of this new method some practical examples have been solved and their solutions are compared with the solution of original diffusion equation. The developed FADM is useful for researchers to think differently and to discover and describe the natural phenomenon (physical, economical, biological, chemical processes) with wonderful tool of mathematics, which is called Fractional Calculus. This paper will generate physical and engineering essence of fractional calculus. Our minor contribution to the vast area of fractional calculus is to handle linear and nonlinear fractional differential equations, to have physical sense to solve that fractional differential equation, to interpret laws of nature in simple way, to make fractional calculus interesting, to give picturesque sense to complex looking mathematics. Furthermore, this research work is useful to encourage new researchers to carry out research and development on scientific and engineering aspects. We organize the paper as follows: In Section 2, some definitions and properties of fractional calculus are presented. Section 3 is devoted to develop the FADM for space fractional partial differential equations. In section 4, we present examples to show the efficiency of ADM. Finally, relevant conclusions are drawn in section 5.

II. PRELIMINARIES AND NOTATIONS

In this section, we study some definitions and properties of fractional calculus.

Definition 2.1 A real function f (t), t > 0 , is said to be in the space C_α , $\alpha \in R$ if there exists a real number $p > \alpha$, such that $f(t) = t^p f_1(t)$, where $f_1(x) \in C[0,\infty)$ and it is said to be in the space $C_\propto^m$ if and only if $f^{(m)}(t) \in C_\alpha$, $m \in N$.

Definition 2.2 The Riemann - Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt \,, \alpha > 0, x > 0$$

$$J^0 f(x) = f(x)$$

Definition 2.3 The Caputo derivative of fractional order α of a function $f(t)$, $f(t) \in C_{-1}^m$ is defined as follows

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(t-\tau)^{(1-m+\alpha)}}\, d\tau, \quad for \quad m - 1 < \alpha \leq m,\ m \in N,\ x > 0,$$

(i) $D_x^\beta Sinx = Sin(x + \frac{\beta\pi}{2})$

(ii) $D_x^\beta Cosx = Cos(x + \frac{\beta\pi}{2})$

Properties:

For $f(x) \in C\mu$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$, we have

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$

It is simple to prove the following properties of fractional derivatives and integrals that will be used in the analysis

(i) $D^\alpha J^\alpha f(t) = f(t)$

(ii) $J^\alpha D_*^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^k(0^+)\frac{t^k}{k!}$, x > 0

(iii) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\, t^{\gamma+\alpha}$

(iv) $D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\, t^{\gamma-\alpha}$

In the next section, we develop the fractional Adomain decomposition method for fractional partial differential equation.

III. THE FRACTIONAL ADOMIAN DECOMPOSITION METHOD (FADM)

In order to elucidate the solution procedure of the FADM, we consider the following general fractional partial differential equation.

$$Lu(x, t) + R^\beta u(x, t) + Nu(x, t) = g(x, t),\ m - 1 \leq \beta \leq m,\ x \in R,\ t > 0 \tag{3.1}$$

where L is the operator of the highest order derivatives, R is fractional order derivative, N is nonlinear operator and g is source term. Let

$$R^\beta = \frac{\partial^{n\beta}}{\partial x^{n\beta}} = D^\beta\, D^\beta\, D^\beta\, ...\, D^\beta \tag{3.2}$$

is the $(n\beta)th$ order fractional derivative then the corresponding $R^{-\beta}$ operator will be written in the following form

$$J^\beta = R^\beta = \frac{1}{\Gamma^n(\beta+1)} \int_0^x \int_0^{\tau_n} \int_0^{\tau_{n-1}} ....\int_0^{\tau_2} (d\tau_1)^\beta\, (d\tau_2)^\beta (d\tau_3)^\beta .....(d\tau_n)^\beta$$

and

$$\frac{1}{\Gamma(\beta+1)} \int_0^x (d\tau_n)^\beta$$

is the Caputo integration.

The solutions can be obtained by using L_t^{-1} or $R_x^{-\beta}$, however using L_t^{-1} t requires the use of initial conditions only whereas operating $R_x^{-\beta}$, imposes the use of initial and boundary conditions. Therefore, to reduce the size of calculations, we apply the decomposition method in t-direction.

Since L_t is a second order differential invertible linear operator, therefore, L_t^{-1} is assumed as an integral operator given by

$$L_t^{-1}u(x, t) = \int_0^t \int_0^t u(x,t)\, dt\,,$$

$$L_t^{-1}Lu(x, t) = u(x, t) - u(x, 0) - tu_t(x, t)$$

Operating with the operator L_t^{-1} on both sides of equation (3.1), we have

$$L_t^{-1}Lu(x,t) + L_t^{-1}[R^\beta u(x,t) + Nu(x,t)] = L_t^{-1} g(x, t),\ m\text{-}1 < \beta \leq m$$

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u(x, t) = u(x, 0) + tu$_t$(x, 0) - $L_t^{-1}[R^{\beta}u(x,t) + Nu(x,t)] + L_t^{-1}g(x,t)$, *m-1 < β ≤ m* (3.3)

Now, we decompose the unknown function $u(x, t)$ into sum of an infinite number of components given by the decomposition series

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) \quad (3.4)$$

The nonlinear terms $Nu(x, t)$ are decomposed in the following form:

$$Nu(x,t)= \sum_{n=0}^{\infty} A_n \quad (3.5)$$

where the Adomian polynomial can be determined as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^nN}{d\lambda^n}\left(\sum_{k=0}^{n}\lambda^k u_k\right)\right]_{\lambda=0} \quad (3.6)$$

where *An* are called Adomian polynomials, which can be calculated easily with the help of mathematica software. Substituting the decomposition series (3.4) and (3.5) into both sides of equation (3.3) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + tu_t(x,0) - L_t^{-1}R^{\beta}\left[\sum_{n=0}^{\infty} u_n(x,t)\right] - L_t^{-1}\left[\sum_{n=0}^{\infty} A_n\right] + L_t^{-1}g(x,t) \quad (3.7)$$

The components $u_n(x, t)$, $n \geq 0$ of the solution $u(x, t)$ can be recursively determined by using the relations as follows:

$$u_0(x,t) = u(x,0) + tu_t(x,0) + L_t^{-1}g(x,t)$$

$$u_1(x,t) = -L_t^{-1}[R^{\beta}u_0(x,t)] - L_t^{-1}A_0$$
$$u_2(x,t) = -L_t^{-1}[R^{\beta}u_1(x,t)] - L_t^{-1}A_1$$
$$u_3(x,t) = -L_t^{-1}[R^{\beta}u_2(x,t)] - L_t^{-1}A_2$$

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$$u_{n+1}(x,t) = -L_t^{-1}[R^{\beta}u_n(x,t)] - L_t^{-1}A_n$$

where each component can be determined by using the preceding components and we can obtain the solution in a series form by calculating the components $u_n(x, t)$, $n \geq 0$. Finally, we approximate the solution $u(x, t)$ by the truncated series.

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$$
$$\lim_{N\to\infty} \phi_N = u(x,t)$$

In the next section, we illustrate some examples and their solutions are represented graphically by mathematica software.

IV. APPLICATIONS

FADM for Space Fractional Partial Differential Equations:

Consider the space fractional partial differential equation

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}} + g(x,t) \quad (4.1)$$

on a finite domain $x_L < x < x_R$ with $1 < \beta \leq 2$ and the diffusion coefficient $d(x) > 0$.

The operator form of (9) can be written as

$$L_t u(x,t) = d(x)D_x^{\beta}u(x,t) + g(x,t) \quad (4.2)$$

Therefore, by FADM we can write

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) + L_t^{-1}\left[d(x)D_x^{\beta}\sum_{n=0}^{\infty} u_n(x,t)\right] + L_t^{-1}g(x,t)$$

then each term of series is given by Adomian Decomposition Method recurrence relation

$$u_0(x,t) = u(x,0) + L_t^{-1}g(x,t),$$
$$u_{n+1}(x,t) = L_t^{-1}\left[d(x)D_x^{\beta}u_n(x,t)\right] \quad (4.3)$$

It is worth noting that once the zeroth component u_0 is defined, and then the remaining components u_n, $n \geq 1$ can be completely determined. Therefore, the series solution is entirely determined.

Test Problem (i): Consider the following space fractional partial differential equation

$D_t\, u(x,t) = D_x^{\beta}u(x,t)$, $0 < x < \pi$, $1 < \beta \leq 2$, $t > 0$

Initial condition: $u(x, 0) = \sin x$

Boundary conditions: $u(0, t) = 0$, $u(\pi, t) = 0$, $t \geq 0$

Using equation (4.3), we have

$$u_0(x,t) = u(x,0) = \sin x$$

$$u_1(x,t) = L_t^{-1}[D_x^{\beta}u_0] = L_t^{-1}[D_x^{\beta}\sin x] = t\sin\left(x+\frac{\beta\pi}{2}\right),$$

$$u_2(x,t) = L_t^{-1}[D_x^{\beta}u_1] = L_t^{-1}\left[D_x^{\beta}\, t\sin\left(x+\frac{\beta\pi}{2}\right)\right] = \frac{t^2}{2!}\sin\left(x+\frac{2\beta\pi}{2}\right),$$

$$u_3(x,t) = L_t^{-1}[D_x^{\beta}u_2] = L_t^{-1}\left[D_x^{\beta}\frac{t^2}{2!}\sin\left(x+\frac{2\beta\pi}{2}\right)\right] = \frac{t^3}{3!}\sin\left(x+\frac{3\beta\pi}{2}\right),$$

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Therefore, the series solution of IBVP is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots\ldots$$

By substituting the above values, we get

$$u(x,t) = \sin x + t\sin\left(x+\frac{\beta\pi}{2}\right) + \frac{t^2}{2!}\sin\left(x+\frac{2\beta\pi}{2}\right) + \frac{t^3}{3!}\sin\left(x+\frac{3\beta\pi}{2}\right) + \ldots$$

$$u(x,t) = \sum_{0}^{\infty}\frac{t^n}{n!}\sin\left(x+\frac{n\beta\pi}{2}\right)$$

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For $\beta = 2$ the exact solution of the original partial differential equation is

$u(x, t) = e^{-t} \sin x$

The graphical representation of the solution is as follows:

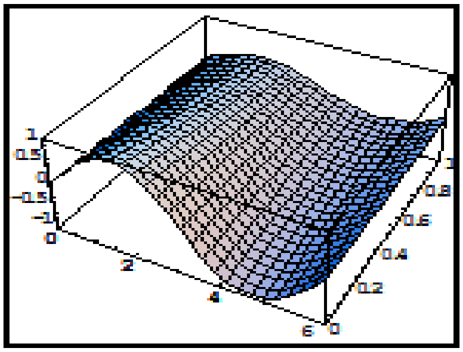

Fig. 4.1: The exact solution of one-dimensional diffusion equation.

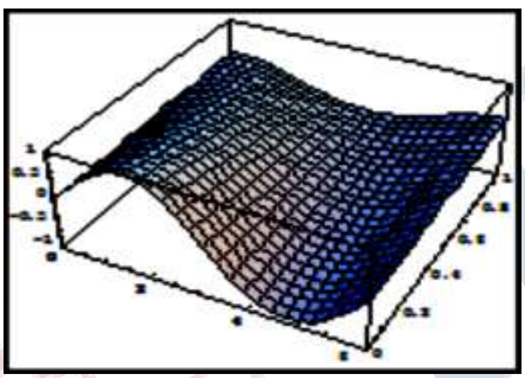

Fig.4.2: Solution for one-dimensional space fractional diffusion equation with $\beta = 1.8$

Test Problem (ii): Consider the following space fractional partial differential equation

$D_t u(x, t) = D_x^{\beta} u(x, t) + \sin x,\ 0 < x < \pi,\ 1 < \beta \leq 2,\ t > 0$

Initial condition: $u(x, 0) = \cos x$

Boundary conditions: $u(0, t) = e^{-t}$, $u(\pi, t) = -e^{-t}$, $t \geq 0$

By using FADM, we have following recursive relation

$u_0(x, t) = u(x, 0) + L_t^{-1} g(x, t),$

$u_0(x, t) = \cos x + L_t^{-1} \sin x$

$u_0(x, t) = \cos x + t \sin x$

$u_1(x, t) = L_t^{-1}[u_0(x, t)]$

$u_1(x, t) = t \cos(x+\beta\pi/2) + \frac{t^2}{2!} \sin(x+\beta\pi/2)$

Similarly we can calculate the values of $u_2(x, t)$ we have

$u_2(x, t) = L_t^{-1}[D_X^{\beta}\, u_1(x, t)],$

$u_2(x, t) = \frac{t^2}{2!}\cos(x+2\beta\pi/2) + \frac{t^3}{3!}\sin(x+2\beta\pi/2)$

.

Therefore, the series solution for the IBVP is given by

$u(x, t) = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \ldots\ldots$

Substituting values of components in above equation, the solution of the original IBVP is given by

$u(x, t) = \cos x + t \sin x + t \cos(x+\frac{\beta\pi}{2}) + \frac{t^2}{2!} \sin(x+\frac{\beta\pi}{2}) + \frac{t^2}{2!}\cos(x+\frac{2\beta\pi}{2}) + \frac{t^3}{3!}\sin(x+\frac{2\beta\pi}{2}) + \ldots.$

$u(x, t) = [\cos x + t\cos(x+\frac{\beta\pi}{2}) + \frac{t^2}{2!}\cos(x+\frac{2\beta\pi}{2}) + \ldots] + [t\sin x + \frac{t^2}{2!}\sin(x+\frac{\beta\pi}{2}) + \frac{t^3}{3!}\sin(x+\frac{2\beta\pi}{2}) + \ldots]$

$u(x, t) = \sum_0^{\infty} \frac{t^n}{n!} \cos(x + \frac{n\beta\pi}{2}) \sum_0^{\infty} \frac{t^{n+1}}{(n+1)!} \sin(x + \frac{n\beta\pi}{2})$

If $\beta = 2$ then the closed form solution of the original IBVP is

$u(x, t) = \cos x\, e^{-t} + \sin x\,(1 - e^{-t})$

The graphical representation of the solution is as follow:

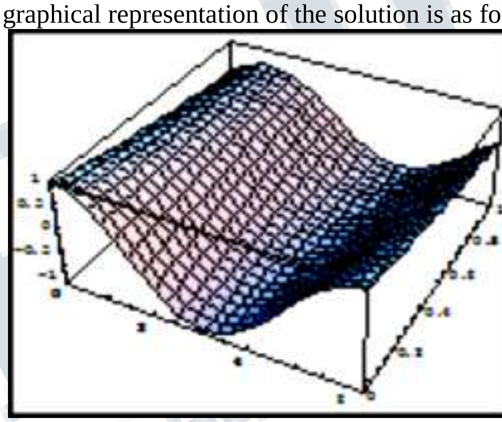

Fig. 4.3: The exact solution of one-dimensional diffusion equation.

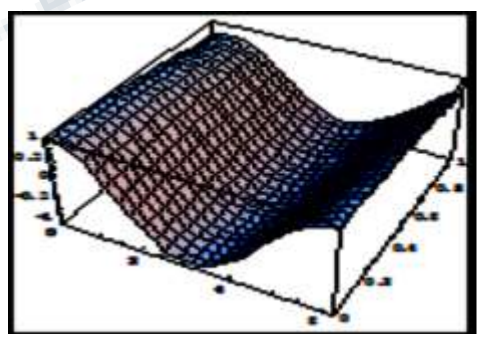

Fig. 4.4: Solution for one-dimensional space fractional diffusion equation with β = 1.8

V. CONCLUSIONS

The main objective of this work is to obtain a solution for space fractional partial differential equations. We observe that Adomian Decomposition Method is a powerful method to solve space fractional partial differential equations. The method is applied to obtain the solutions of several examples to show the

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applicability and efficiency of the proposed method. The obtained results demonstrate the reliability of the algorithm. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods. Finally, we come to the conclusion that the ADM is very powerful and efficient in finding solutions for wide class of space fractional partial differential equations.

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